Table of contents

Insurance 101

Insurance is an effective risk management tool used to protect against contingent losses of market participants.

$$X$$

$$\underbrace{I(X)}_{\text{Reimbursement}} \qquad \underbrace{X - I(X)}_{\text{Insured's retained loss}}$$

$$\underbrace{I(X) - \underbrace{\pi(I(X))}_{\text{premium}}}_{\text{Insurer's loss or benefit}} \qquad \underbrace{X - I(X) + \underbrace{\pi(I(X))}_{\text{premium}}}_{\text{Insured's total loss}}$$

where $I \in \mathcal{I}$ is an admissible indemnity function, and π is a premium principle.

Classical optimization problems in insurance

Popular optimal (re-)insurance design problems:

1. Maximize expected utility:

$$\max_{I \in \mathcal{I}} E\left[v(w - X + I(X) - \pi(I(X)))\right] .$$

   Arrow (1963): optimality of a stop-loss contract.
   Gerber(1979), Young (1999), Kaluszka (2001,2005), etc.

2. Minimize risk measure:

$$\min_{I \in \mathcal{I}} \rho(X - I(X) + \pi(I(X))) .$$

   Cai et al. (2008), Kaluszka and Okolewki (2008), Bernard and Tian (2009), Cheung (2010), etc.

All problems are considered under the assumption that the distribution of X is known. Can we take this assumption for granted?

Uncertainty

- From data to models
  - Parameter uncertainty
    Estimation error, simulation error, etc
  - Model uncertainty
    Choice of models, complexity of models, etc.

- Distributional uncertainty
  - Only partial information about the true distribution are observed from the historical data.
  - Changes of the underlying risks
  - In a conservative decision, the worst-case distribution is important

Worst-case scenario

Suppose an agent faces an underlying risk X

- ℓ is the loss function/strategy the agent adopts.
- ρ is the risk measure used to quantify the agent's risk exposure
- $\mathcal{S}$ is the uncertainty set includes all distributions of alternative risks considered

From the perspective of risk management, the worst-case scenario in which the agent has the largest risk exposure is of special interests.

The agent's optimization problem with model uncertainty can be formulated as

$$\min_{\ell} \underbrace{\sup_{F \in \mathcal{S}} \rho(\ell(X^F))}_{\text{worst-case scenario}}, \quad X^F \sim F.$$

Literature

In the literature of insurance

- Asimit et al. (2017): for $= \mathrm{VaR}, \mathrm{ES}$,

$$\begin{cases} \min_{(I,P)\in I} \max_{R\, k\in M} \{ {}_{P_k}(X \quad I(X) + \quad P) \}; \\ \text{s.t. } \omega_0 + (1 + \quad) H_{P_k}(I(X)) \quad P \quad P; \forall k \in M \; : \end{cases}$$

where P_k, $k \in M$ includes finite many probability measures.

- Birghila and Pflug (2019)

$$\min_{I\in I} \max_{F\in C} \{ (X^F \quad I(X^F) + \quad (I(X^F)) \}; \quad \text{s.t.} \quad (I(X^F)) \quad B$$

where C is the convex cone of n reference distributions.

- Liu and Mao (2021): for $= \mathrm{VaR}, \mathrm{ES}$,

$$\min_{d \quad 0} \sup_{F\in S(\;;\;)} (X^F \wedge d + (1 + \quad) E^F[(X^F \quad d)_+]) :$$

where $S(\;;\;)$ gives first & second moments constraints.

In this talk, we focus on the worst-case scenario for an agent

$$\sup_{F\in S} \rho_h(\ell(X^F)), \qquad X^F \sim F$$

where

- ρ_h is a distortion risk measure (e.g. Dhaene et al. (2012)):

$$\rho_h(X^F) = \int_{-\infty}^{0} -h(F(x))\,dx + \int_{0}^{\infty} 1 - h(F(x))\,dx = \int_0^1 \phi(u)F^{-1}(u)\,du;$$

  where $h : [0,1] \mapsto [0,1]$ is non-decreasing (convex) with $h(0) = 0$ and $h(1) = 1$, and $\phi(u) = h'(u)$, $0 < u < 1$
- S is the uncertainty set defined by Wasserstein distance constraints
- ℓ is the loss function/strategy the agent adopts.

Uncertainty set with Wasserstein distance constraint

For $X \sim F$ and $Y \sim G$, for $k \ge 1$, the Wasserstein distance is

$$W_k(X;Y) = W_k(F;G) = \left(\int_0^1 \left| F^{-1}(x) - G^{-1}(x) \right|^k dx \right)^{1/k}.$$

The uncertainty set with Wasserstein distance constraint

$$S = \{\text{r.v. } Y : W_k(Y;X) \le \varepsilon\}$$ "W97 0 87luel)

Uncertainty set with Wasserstein distance constraint

Theorem (Proposition 4 in Liu et al. (2022))

For a continuous and convex distortion function h,

$$\sup\left\{\rho_h(X^G) : W_k(G,F) \le \varepsilon\right\} = \rho_h(X^F) + \varepsilon\|\gamma\|_q,$$

where $q = (1 - 1/k)^{-1}$ with the convention $0^{-1} = \infty$, and $\|\cdot\|_q$ is the L_q-norm.

For $k > 1$, the above maximum value is attained by the worst-case distribution

$$G^{-1}(t) = F^{-1}(t) + \varepsilon\frac{(\gamma(t))^{q-1}}{\|\gamma\|_q^{q/k}}, \qquad 0 < t < 1.$$

Example { Expected shortfall (ES)

Take = ES for 2 $(0;1)$, then $(X) = \int_0^1 \mathrm{VaR}_t(X)\,dh(t)$, where

$$h(t) = \frac{1}{1}(t \quad)^+ \quad \text{and} \quad (t) = \frac{1}{1} 1_{[\;;\;1]}:$$

The worst-case value is

$$\sup \; \mathrm{ES}\;(X^G) : W_k(G;F$$

Uncertainty set with Wasserstein distance constraint

Uncertainty set is

$$S = \{G : W_k(G;F) \leq \varepsilon\}$$

where $X^F \sim F$ is considered as a reference distribution, and ε is the tolerant bound for the Wasserstein distance.

Consider the worst-case scenario:

$$\sup_{G \in S} \rho_h(\ell(X^G)) = \sup \{\rho_h(\ell(X^G)) ; W_k(G;F) \leq \varepsilon\} ;$$

with two types of loss functions:

- Stop-loss function: (optimal to the utility maximization)

$$\ell(x) = (x - d)^+$$

- Limited-loss function: (optimal to the VaR minimization)

$$\ell(x) = \min\{x ; M\}$$

Stop-loss function

Take $`_1(x) = (x \quad d)^+$ for $d >$ ess-inf(X)

Worst-case risk measure

$$\sup \quad _h((X^G \quad d)^+) : W_k(G; F) \quad "$$

For $\quad 2\ [0;1]$, de ne $\ _{1;} \ := \quad I_{[\ ;\ 1]}$ which is again a non-negative and increasing function.

$$\sup_{G2S} \ _h \ (X^G \quad d)^+ = \sup_{G2S} \int_{G(0)}^{1} \quad (u) \ \ G^{\ 1}(u) \quad d \ \ du$$

$$= \sup_{G2S} \max_{2[0;1]} \int^{1} \quad (u) \ \ G^{\ 1}(u) \quad d \ \ du$$

$$= \sup_{2[0;1]} \underbrace{\sup_{G2S} \int_0^1 \ _{1;}(u) \ \ G^{\ 1}(u) \quad d \ \ du}_{\text{worst-case without transform}};$$

Wasserstein distance constraint and stop-loss transform

Theorem (Cai et al. (2022b))

Take k   1 and q = (1   1=k) 1.

(i) The worst-case risk measures value is

$$\sup \left\{ \ _h((X^G \quad d)^+) : W_k(G;F) \quad " \right\}$$

$$= \max_{2[0;1]} \int_0^1 \ _{1;} \ (u)F^{\ 1}(u)du + "k \ _{1;} \ k_q \quad dk \ _{1;} \ k_1 \quad :$$

(ii) The worst-case distribution is given by

$$G^{\ 1}(t) = F^{\ 1}(t) + " \frac{(\ _{1;} \ (t))^{q \ 1}}{k \ _{1;} \ k_q^{q=k}}; \quad 0 < t < 1:$$

where   is the maximizer in (i).

Example - Expected shortfall

Take $= \mathrm{ES}$ for some $2\ (0;1)$.

(i) The worst-case value is

$$\sup \ \overset{n}{} \mathrm{ES}\ ((X^{G} \quad d)^{+}) : W_{k}(G;F) \quad " \overset{o}{}$$

$$= \frac{1}{1} \max_{2[\ ;\ 1]} \overset{n}{} (1 \quad)\ \mathrm{ES}\ (X^{\hat{F}}) \quad d \ + "(1 \quad)^{1=k} \overset{o}{} :$$

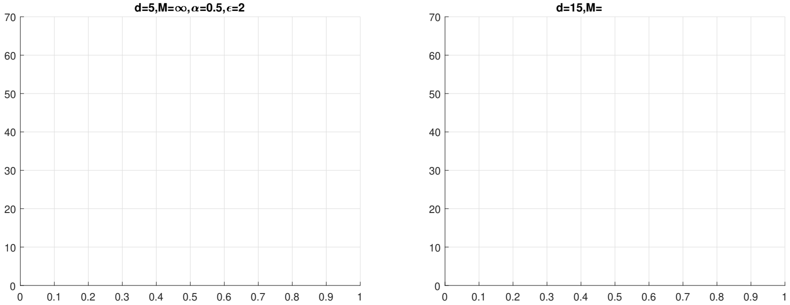
(ii) The worst-case distribution is

$$G^{\ 1}(t) = F^{\ 1}(t) + " \ \frac{(\ _{1;} \ (t))^{q \ 1}}{k \ _{1;} \ k_{q}^{q=k}}$$

where $_{1;} \ = \frac{1}{1} I_{[\ \_ \ ;1]}$ and is the solution to the maximization problem in (i).

Example - Wang's premium

Figure: Worst-case distributions with stop-loss function.

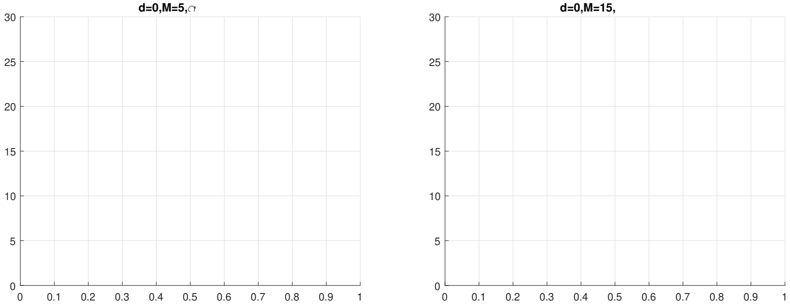


Limited-loss function

Take $\ell_2(x$

Example - Wang's premium (cont')

Figure: Worst-case distributions with limited loss function.



Wasserstein distance constraint and limited stop-loss transform

- Wang's premium ρ_h with $h(u) = 1 - \Phi(\Phi^{-1}(1-u) + 0.5)$.
- Exponential reference $F_1(x) = 1 - e^{-x/4}$, $x \geq 0$
- Pareto reference $F_2(x) = 1 - \left(\frac{12}{x+12}\right)^4$
- Limited stop-loss function

$$\ell(x) = \max\{(x - d)^+; M\}$$

- Wang's premium in the worst-case:

$$\sup_G \rho_h\left(\max\{(X^G - d)^+; M\}\right); W_2(G; F_i) \leq \varepsilon; \quad i = 1; 2.$$

Isotonic Projection: For $h \in L^2(0,1)$, let

$$h^{\uparrow} = \arg\min_{k \in K} \| h - k \|^2,$$

$$\text{where } K = \left\{ k : (0,1) \mapsto \mathbb{R} \;\; \int_0^1 k(u)^2 du < \infty, \; k \text{ non-decreasing} \right\}.$$

Notation

Denote $_{1,}(u) := (u)\, 1_{[\,,1]}(u)$, for $u \in [0,1]$, and the isotonic Projection for $_{1,} + F^{-1}$ for some ≥ 0 as

$$h^{\uparrow}_{1,,} = \arg\min_{h \in K} \| h - _{1,} - F^{-1} \|_2.$$

Denote $_{2,}(u) := (u)\, 1_{[0,\,]}(u)$, for $u \in [0,1]$, and the isotonic Projection for $_{2,} + F^{-1}$ for some ≥ 0 as

$$h^{\uparrow}_{2,,} = \arg\min_{h \in K} \| h - _{2,} - F^{-1} \|_2.$$

Wasserstein distance plus moments constraints and stop-loss transform

Theorem (Cai et al. (2022a))

Consider the worst-case problem $\sup_{G \in S} \; h \; (Y^G \quad d)_+ \; :$
The quantile function of the worst-case distribution is

$$G^{\;1}(u) = \quad + \quad \left(\frac{h^{"}_{1;\;\;;}(u) \quad a_{\;;}}{b_{\;;}} \right) ; \quad 0 < u < 1;$$

where $a_{\;;} = E[h^{"}_{1;\;\;;}(U)]$, $b_{\;;} = \sqrt{\mathrm{var}(h^{"}_{1;\;\;;}(U))}$, $\quad > 0$ is determined uniquely by the distance constraint $W_2(F; G \quad) = \; "$, and

$$= \arg\max_{\in [0;1]} \int_0^1 \; _{1;}(u) \; G^{\;1}(u) \quad d \; du:$$

Example { Expected shortfall

Assume the reference distribution is $F(x) = 1 \quad e^{\,x=5}$, $= \quad = 5$,
" = 1, and $_h$ = $ES_{0:9}$.ssume3. Wors9c28.909 1(0)]14R8= 1

Wasserstein distance plus moments constraints and limited-loss transform

Theorem (Cai et al. (2022a))

Consider the worst-case problem $\sup_{G \in S} h \; Y^{G} \wedge M$:
The quantile function of the worst-case distribution is

Example { Expected shortfall

Assume the reference distribution is $F(x) = 1 \quad e^{\ x=5}$, $=\quad = 5$, $" = 1$, and $_h = ES_{0:9}$:

d	
10	[0; 0:9]
20	0:9835

Summary

- In this talk we discuss multiple model uncertainty models
  - Distortion risk measure
  - With or without transform
    - Stop-loss, limited-loss
  - Wasserstein distance, moments contraints

- Future works
  - Other risk measures
  - General transformation
  - Various uncertainty sets: likelihood ratio, KL-divergent, etc.
  - Novel techniques to characterize worst-case distribution and worst-case risk measure value

Reference II

Dhaene, J., Kukush, A., Linders, D., and Tang, Q. (2012). Remarks on quantiles and distortion risk measures. European Actuarial Journal, 2(2):319{328.

El Ghaoui L, Oks M, Oustry F (2003) Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. Oper. Res. 51(4):543{556.

Li, J. Y.-M. (2018). Closed-form solutions for worst-case law invariant risk measures with application to robust portfolio optimization. Operations Research, 66(6):1533{1541.

Liu, H. and Mao, T. (2021). Distributional robust reinsurance with value-at-risk and conditional value-at-risk. Available at SSRN 3849078

Liu, F. and Wang, R. (2021). A theory for measures of tail risk. Mathematics of Operations Research, 46(3), 1109{1128.

Hu, X., Yang, H., and Zhang, L. (2015). Optimal retention for a stop-loss reinsurance with incomplete information. Insurance: Mathematics and Economics, 65:15{21.

Sion, M. (1958). On general minimax theorems, Paci c Journal of mathematics , 8(1), 171{176.